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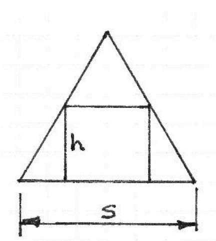
**Advanced Precalculus**
**Expressing One Variable in terms of Others (Quantitative Reasoning)**

Your ability to express one variable in terms of others is of critical importance. To express one variable in terms of others you have four tools:

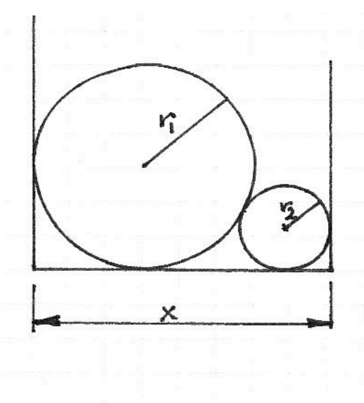
1. Algebraic relationships.
2. Geometric relationships.
3. Trigonometric relationships.
4. Functional relationships.

Practice Problems

1. The equilateral triangle shown has side length s. The inscribed rectangle has height *h*. Express the area of the inscribed rectangle as a function of *s* and *h*.



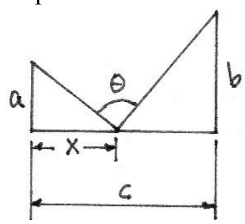

2. Express $x$ in terms of $r_1$ and $r_2$.



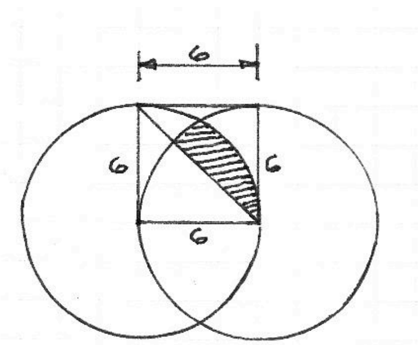

3. A soup company packages soup in cans with volume 375 $cm^3$ and surface area 293.9 $cm^2$.
   a. Find a polynomial function for the radius, *r*. (It is a cubic.)
   b. Use your grapher to find solutions for *r*.
   c. Use the largest of your positive solutions for *r* and determine the height of the can, *h*.

4. This problem is very difficult. An underground gasoline storage tank is a horizontal cylinder of length, *L*, and radius, *r*. The amount of fuel in the tank is determine by inserting a measuring stick into the tank from the top and measuring how deep the liquid is, *d*, in the tank.
   a. Find an equation for the volume of the liquid in the tank in terms of *r*, *L*, and *d*. For simplicity, assume the tank is less than half full.
   b. Assume that the tank is 24 feet long with a radius of 4 feet. Determine the volume of fuel in the tank if the measuring stick shows the liquid to be 3 feet deep. Convert your answer into gallons using the conversion factor 7.48 gal/$ft^3$.

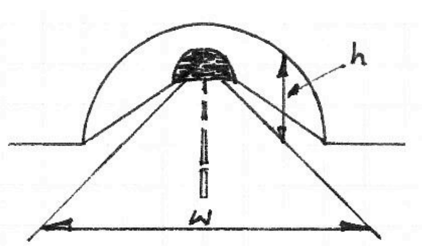
5. Express θ as a function of a, b, c, and x.




6. Determine the area of the shaded region.




7. The tunnel opening depicted below is a semicircle. The road is width *w*, and the minimum clearance at the edge of the road is *h*. Express the maximum height at the center of the tunnel, which we will call *M*, in terms of *h* and *w*.




8. Repeat the previous problem, given that the tunnel opening is a parabola and the roadway is right on the focus. Express the maximum height at the center of the tunnel, which we will call *M*, in terms of *h* and *w*.

9. This problem was on the 2011 PSAT:
   At a company, there are *n* more male employees than female employees. If there are *k* male employees at the company, what fraction of the employees are male, in terms of *n* and *k*?

10. This problem was on the 2011 PSAT:
    A right circular cylinder has a radius *r* and a height *h*. Not including its bases, the surface area of the cylinder is $2\pi rh$. If this surface area not including the bases is $70\pi$, what is the volume of the cylinder in terms of *r*?

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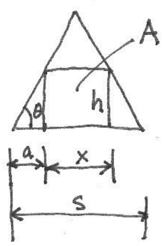
**Solutions**

1.




$\theta = 60^\circ$, so $\tan\theta = \frac{h}{a} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

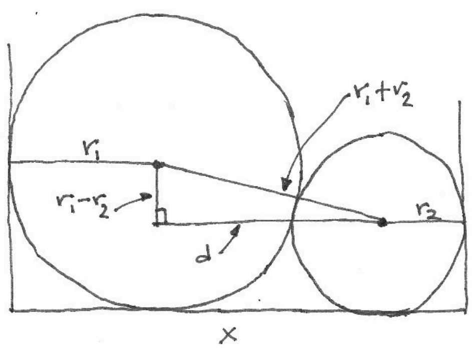
Thus, $a = \frac{h}{\sqrt{3}}$

$s = x + 2a$

$x = s - 2a = s - \frac{2h}{\sqrt{3}}$

$$\boxed{A = x \cdot h = \left(s - \frac{2h}{\sqrt{3}}\right)h}$$

2.




$x = r_1 + r_2 + d$

$d^2 = (r_1 + r_2)^2 - (r_1 - r_2)^2$

$= r_1^2 + 2r_1r_2 + r_2^2 - (r_1^2 - 2r_1r_2 + r_2^2)$

$= 4r_1r_2$

$d = 2\sqrt{r_1r_2}$

$$\boxed{x = r_1 + r_2 + 2\sqrt{r_1r_2}}$$

3. $V = 375 \text{ cm}^3$

$\pi r^2 h = 375$

$h = \frac{375}{\pi r^2}$ $\longrightarrow$

$S = 293.9 \text{ cm}^2$

$2\pi r h + 2\pi r^2 = 293.9$

$2\pi r\left(\frac{375}{\pi r^2}\right) + 2\pi r^2 = 293.9$

Multiply by r, cancel, & gather terms:

$6.28r^3 - 293.9r + 750 = 0$

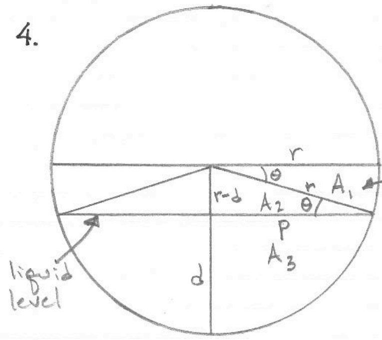
Using a grapher, the solutions to this are

$r = -7.9$ cm, $r = 3.4$ cm, $\boxed{r = 4.5 \text{ cm}}$

The first of these is unrealistic. The other two can work. Using the larger radius,

$h = \frac{375}{\pi r^2} = \frac{375}{\pi(4.5)^2} = \boxed{5.9 \text{ cm}}$

4.




$\underline{\underline{A_1}}$

$A_1 = \frac{1}{2}r^2\theta$ but θ relates to d since

$\sin\theta = \frac{r-d}{r}$, so $\theta = \sin^{-1}\left(\frac{r-d}{r}\right)$

$A_1 = \frac{1}{2}r^2\sin^{-1}\left(\frac{r-d}{r}\right)$

$2A_1 = r^2\sin^{-1}\left(\frac{r-d}{r}\right)$

$\underline{\underline{A_2}}$

$A_2 = \frac{1}{2}(r-d)(p)$ But p relates to r & d by the Pythagorean theorem

$(r-d)^2 + p^2 = r^2$

$p^2 = r^2 - (r-d)^2 = r^2 - r^2 + 2rd - d^2 = 2rd - d^2$

$p = \sqrt{2rd - d^2}$

$A_2 = \frac{1}{2}(r-d)\sqrt{2rd-d^2}$

$2A_2 = (r-d)\sqrt{2rd-d^2}$

$\underline{\underline{A_3}}$

$A_3 = \frac{1}{4}(\pi r^2) - A_1 - A_2$

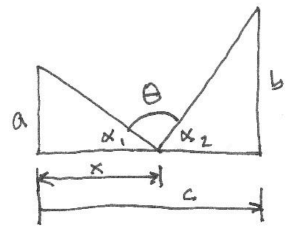
$2A_3 = \frac{1}{2}\pi r^2 - 2A_1 - 2A_2 = \boxed{\frac{1}{2}\pi r^2 - r^2\sin^{-1}\left(\frac{r-d}{r}\right) - (r-d)\sqrt{2rd-d^2}}$

Volume of liquid in tank is $2A_3 \cdot L$

$V = L\left(\frac{1}{2}\pi r^2 - r^2\sin^{-1}\left(\frac{r-d}{r}\right) - (r-d)\sqrt{2rd-d^2}\right)$

For $L = 24$, $r = 4$, $d = 3$ $V = 413.2\ ft^3 \cdot 7.48\,\frac{gal}{ft} = 3090.8\,gal.$

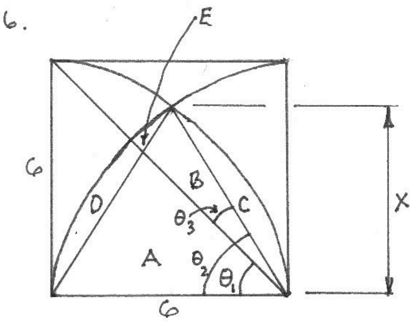
5.




$\alpha_1 = \tan^{-1}\frac{a}{x}$ $\quad$ $\alpha_2 = \tan^{-1}\frac{b}{c-x}$

$\theta\,(\text{deg}) = 180 - \tan^{-1}\frac{a}{x} - \tan^{-1}\frac{b}{c-x}$

6.




Solution = C + B + E

$\theta_1 = 45^\circ$ $\quad$ $\theta_2 = 60^\circ$ $\quad$ so $\theta_3 = 25^\circ$

$\theta_3 = 25^\circ \cdot \frac{2\pi}{360^\circ} = \frac{5\pi}{36}$

Sector Area $= \frac{1}{2}R^2\theta$

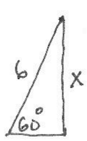
Thus E + B (a sector w/ rad = 6)

$= \frac{1}{2}\cdot 36 \cdot \frac{5\pi}{36} = \frac{5\pi}{2}$

A + B + C forms another sector of radius 6 and $\theta = \frac{\pi}{3}$

$A+B+C = \frac{1}{2}\cdot 36 \cdot \frac{\pi}{3} = 6\pi$

A + B is an equilateral triangle, so $A+B = \frac{1}{2}\cdot 6 \cdot x = 3x$




$\sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{x}{6}$, $\Rightarrow$ $x = 3\sqrt{3}$

$A+B = 9\sqrt{3}$, and $C = 6\pi - 9\sqrt{3}$

$C+B+E = \frac{5\pi}{2} + 6\pi - 9\sqrt{3} = \boxed{\frac{17\pi}{2} - 9\sqrt{3}}$ or $\frac{17\pi - 18\sqrt{3}}{2}$

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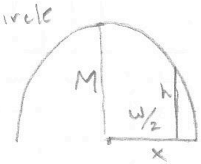
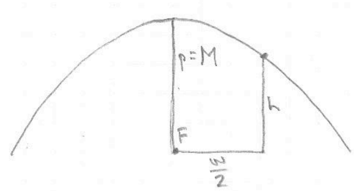
7. Semicircle




$$x^2+y^2=M^2 \qquad \left(\frac{w}{2}\right)^2+h^2=M^2$$

$$M=\sqrt{\frac{w^2}{4}+h^2}$$

8. Parabola




$$x^2=-4p(y-p)$$

$$\left(\frac{w}{2}\right)^2=-4M(h-M)$$

$$w^2=M^2-Mh$$

$$M^2-Mh-w^2=0$$

$$M=\frac{h\pm\sqrt{h^2+4w^2}}{2}$$

9. males = $k$ Fraction male = $\frac{k}{2k-n}$

females = $k-n$

total = $k+k-n = 2k-n$

10. $A = 70\pi = 2\pi rh$

$r=\frac{35}{h}$ $h=\frac{35}{r}$

~~$A = 70\pi$~~ $V=\pi r^2 h = \pi r^2\cdot\frac{35}{r} = \underline{35\pi r}$